Linear Algebra II

05/03/2018, Monday, 14:00 – 16:00

1 $(5+5+5+5=20$ pts) **Inner product spaces**

Let V be a real vector space.

(a) Write down the definition of inner product on V.

(b) Given an inner product on V, write down what the Cauchy-Schwarz inequality says about the inner product of two given vectors in V.

(c) Write down the definition of norm on V.

(d) Assume that $\langle u, v\rangle$ is an inner product on V. Prove that $\|u\| := \sqrt{\langle u, u\rangle}$ is a norm on V.

2 $(10+10=20$ pts) **Gram-Schmidt orthogonalization**

Consider the inner product space $C[0,1]$ with inner product

$$(f, g) := \int_0^1 f(x)g(x)dx.$$

Let $\mathcal{S}$ be the subspace of all functions of the form $g(x) = a + b\sqrt{x}$ with $a, b \in \mathbb{R}$.

(a) Determine an orthonormal basis of $\mathcal{S}$.

(b) Compute the best least squares approximation of the function $f(x) = x$ by a function from the subspace $\mathcal{S}$.

3 $(5+5+5+5+5 = 25$ pts) **Eigenvalues**

Let A be a real $n \times n$ matrix, let $v_1, v_2, \ldots, v_k$ be linearly independent vectors in $\mathbb{R}^n$ and define an $n \times k$ matrix V by $V := (v_1\ v_2\ \ldots\ v_k)$ and a subspace $\mathcal{V}$ by $\mathcal{V} := \text{span}(v_1, v_2, \ldots, v_k)$. Let B be a real $k \times k$ matrix such that $AV = VB$.

(a) Prove that $\mathcal{V}$ is A-invariant, i.e. $Av \in \mathcal{V}$ for all $v \in \mathcal{V}$.

(b) Prove that every eigenvalue of B is an eigenvalue of A.

(c) Prove that if A is nonsingular then also B is nonsingular.

(d) Assume $v_1, v_2, \ldots, v_k$ are eigenvectors of A. What does this say about B?

(e) Assume that $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal set. Prove that if A is symmetric then B is symmetric.

4 $(5+5+5+5+5 = 25$ pts) **Hermitian matrices**

A matrix $A \in \mathbb{C}^{n \times n}$ is called a skew-Hermitian matrix if $A^H = -A$.

(a) Show that if A is skew-Hermitian then $\text{Re}(x^H A x) = 0$ for all $x \in \mathbb{C}^n$.

(b) Show that if A is skew-Hermitian then for every eigenvalue λ of A we have $\text{Re}(\lambda) = 0$.

(c) Show that if A is skew-Hermitian then A is unitarily diagonalizable, that is, there exists a unitary matrix U such that $U^H A U$ is a diagonal matrix

(d) Show that if $U \in \mathbb{C}^{n \times n}$ is unitary, then for every eigenvalue λ of U we have $|\lambda| = 1$.

(e) Assume now that U is unitary and skew-Hermitian. What does this say about the eigenvalues of U?

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10 pts free